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The role of drift mass in the kinetic energy and momentum of periodic water waves and sound waves

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For two-dimensional periodic water waves or sound waves, the kinetic energy per wavelength is $\frac{1}{2}m_d c^2$, and the momentum per wavelength is $\pm m_d c$, where c is the wave velocity, and m_d is the drift mass per wavelength. These results also hold for three-dimensional periodic waves, for which the kinetic energy, momentum, and drift mass are all for one wave cell, the area of which is the product of the wavelengths in two perpendicular directions.

The results obtained are rigorous, and not restricted to linear waves or even to nonlinear symmetric waves. For linear water waves, in particular, the kinetic energy can be shown to be equal to the sum of the potential energy and the surface energy (due to surface tension), so that the total energy E is twice the kinetic energy, and

$$E = m_d c^2.$$

McIntyre's (1981) contention that wave momentum is a myth is discussed at length for both water waves and sound waves.

1. Introduction

Since this article has to do with the kinetic energy, the momentum, and the drift mass of periodic irrotational water waves and sound waves, it is appropriate to recall briefly what is known about these subjects that has a close bearing on the results given herein.

Early in this century Levi-Civita (1912, 1921) showed that, for periodic water waves, twice the kinetic energy per wavelength is equal to the momentum per wavelength times the wave velocity c. But he did not relate either quantity to the drift mass, although drift in water waves was not unknown in his time. Midway through this century and in the same year, Ursell (1953) and Longuet-Higgins (1953) studied mass transport in periodic water waves, and Darwin (1953) studied fluid drift caused by a body moving (along the x-axis) with constant velocity from $x = -\infty$ to $x = \infty$ in an inviscid fluid of constant density and infinite extent. But neither Ursell nor Longuet-Higgins conceived the notion of drift mass per period (in time) of the waves, and Darwin already knew that his drift mass is ill defined, since the defining integral is not uniformly convergent, but is dependent on how infinity is approached. Further studies of Darwin's problem can be found in Yih (1985, 1995) and Eames, Belcher & Hunt (1994). In this article we shall show that the drift mass per period is well defined for water waves and sound waves, and that this mass is simply related to the kinetic energy and momentum per wavelength. The method used is the same as that used in Yih's papers, but the problem is physically different from Darwin's.

In an especially enjoyable volume (vol. 106) of this *Journal*, McIntyre (1981) wrote interestingly about the momentum of sound waves and waver waves. His quotation of

Lewis Carroll† has, as it should, an intimate bearing on the thesis of his paper, which is clearly stated in its title: on the 'wave momentum' myth. This article recounted the controversy regarding sound-wave momentum between Rayleigh and others on the one hand, and Brillouin and others (including McIntyre) on the other. The reader is referred to McIntyre's article for references on this controversy. Suffice it here to say that the former group believed sound waves have momentum and the latter group believed sound waves have only momentum flux, but not momentum. McIntyre gave examples on sound waves and water waves to support the belief of his group.

Clearly, if there is no wave momentum in all contexts and circumstances, Levi-Civita's result (1912, 1921) would lead to the absurdity that a positive quantity (the kinetic energy) is equal to zero, and (as will be shown in the next section) there would be no drift mass and therefore no mass transport, in contradiction to the work of Ursell and Longuet-Higgins. Since the present work generally agrees with the work of all these authors, it is incumbent upon me to explain that wave momentum need not be zero, and even if circumstances are such that it is, the non-zero wave momentum obtained herein will still retain its significance. I shall assume this burden after my results have been presented.

2. Two-dimensional periodic water waves

Consider an infinite train of two-dimensional periodic waves propagating with speed c in the direction of decreasing x in water of depth h, which can be infinite. The amplitude of the waves can be finite, and the waves do not have to be symmetric with respect to their crests. The flow is assumed irrotational, and in a frame moving with the waves the velocity potential ϕ is

$$\phi = cx + \phi'(x, y), \tag{1}$$

where y is measured vertically upward, and $\phi'(x,y) = \phi'(x'+ct,\ y)$, with x' and y denoting the coordinates in a stationary frame of reference and t denoting time.

For symmetric waves, such as the waves treated by Stokes (1847) and Struik (1926),

$$\phi'(0,y) = \phi'(\lambda, y) = 0, \tag{2}$$

where λ is the wavelength, if x is measured from the position of a crest. But (2) is not necessary. All we need is

$$\phi'(0,y) = \phi'(\lambda, y), \tag{3}$$

which applies not only to the familiar symmetric waves, but also to the unsymmetric ones discovered by Chen & Saffman (1980) and investigated by others, as cited in Benjamin's recent paper (1995). All we require is strict periodicity. Then (3) gives

$$\phi(0,y) = \phi(\lambda, y) - c\lambda. \tag{4}$$

Now consider the domain D bounded by $x = 0$, $x = \lambda$, the flat bottom, and the free surface. We have

$$(u, v) = (\phi_x, \phi_y), \quad (u', v') = (\phi'_x, \phi'_y), \tag{5}$$

where (u, v) is the velocity in the moving frame of reference, and (u', v') is the velocity in the frame at rest. Thus

$$u = c + u', \quad v = v'. \tag{6}$$

† The British seem to be perennially fascinated by this author. Other Cantabrigians H. and B. S. Jeffreys (1956) cited him under the titles of three of the twenty five chapters of their book. No other author was cited more than once this way.

The kinetic energy per wavelength is, with ρ denoting the density,

$$\mathrm{KE} = \frac{1}{2}\iint \rho(u'^2+v'^2)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{2}\iint \rho[(u-c)^2+v^2]\,\mathrm{d}x\,\mathrm{d}y = \tfrac{1}{2}I, \tag{7}$$

which defines I. It is clear that

$$I = c^2 I_1 - I_2, \tag{8}$$

where

$$I_1 = \iint \rho\left(1-\frac{u}{c}\right)\mathrm{d}x\,\mathrm{d}y, \quad I_2 = \iint \rho(cu-q^2)\,\mathrm{d}x\,\mathrm{d}y, \tag{9}$$

with

$$q^2 = u^2+v^2. \tag{10}$$

All integrations are carried out in D. For two-dimensional waves, 'per unit width along a crest' is always implied regarding kinetic energy, momentum, etc.

We now seek the meaning of I_1. Obviously $-cI_1$ is the momentum M per wavelength. But there is another, purely kinematic, meaning. The best way of seeing it is to consider the mass m_d (per unit width along a crest) discharged in one period at any given x', say $x' = 0$, in the frame at rest. Obviously

$$-m_d = \int_0^T \rho\left[\int u'\,\mathrm{d}y\right]\mathrm{d}t, \tag{11}$$

where $T = \lambda/c$ is the period, and u' is a function of y and t, and the minus sign is provided in anticipation that the integral is negative. The limits of the inner integral are the bottom and the free surface. But

$$x-ct = x' = 0, \quad c\,\mathrm{d}t = \mathrm{d}x, \tag{12}$$

so that (12) can be written as

$$-m_d = \frac{1}{c}\int\left[\int_0^\lambda \rho u'\,\mathrm{d}x\right]\mathrm{d}y = \iint \rho\frac{u'}{c}\mathrm{d}x\,\mathrm{d}y = -I_1, \quad \text{or} \quad m_d = I_1. \tag{13}$$

In (13), u' has been considered a function of x and y. It will presently be shown that I_1 is positive. Thus the integral in (13) is negative, indicating a drift to the left.

Since, as mentioned before, $-cI_1$ is the momentum M, we have, from (13),

$$M = -m_d\,c, \tag{14}$$

the negative sign indicating that the momentum is to the left, that is, in the direction of propagation. (For waves travelling to the right, the minus sign should be dropped.)

As for the integral I_2 in (6), we can write

$$I_2 = \iint \rho\left(\frac{cu}{q^2}-1\right)\mathrm{d}\phi\,\mathrm{d}\psi, \tag{15}$$

since

$$q^2 = \frac{\partial(\phi,\psi)}{\partial(x,y)}, \tag{16}$$

where ψ is the stream function and the harmonic conjugate of ϕ. But

$$\frac{u}{q^2} = \frac{\partial x}{\partial \phi},$$

so that

$$I_2 = 0, \tag{17}$$

by virtue of (4). Thus (7), (8), (14) and (17) give

$$\mathrm{KE} = \tfrac{1}{2} m_d c^2. \tag{18}$$

Equations (15) and (18) say that m_d plays the role of a momentum mass m_m and of a kinetic-energy mass m_k, or

$$m_d = m_m = m_k. \tag{19}$$

We note that with (17), (8) becomes $I = c^2 I_1$, which was a result of Levi-Civita (1912, 1921). The proof of (17) is considerably shorter than Levi-Civita's proof.

3. Three-dimensional periodic water waves

For three-dimensional waves we can still use (1), but now ϕ and ϕ' depend on x, y, and the third Cartesian coordinate z, measured horizontally in a direction normal to the x-axis. For periodic waves propagating in the direction of (decreasing) x, again we assume, in the moving frame of reference

$$\phi'(0, y, z) = \phi'(\lambda_x, y, z), \tag{20}$$

where λ_x is the wavelength in the x-direction. This can be written as

$$\phi'(0, y, z) = \phi(\lambda_x, y, z) - c\lambda_x. \tag{21}$$

The domain of integration is now bounded by the planes $x = 0$ and $x = \lambda_x$, the flat bottom, the free surface, and the planes $z = 0$ and $z = \lambda_z$, where λ_z is the wavelength in the z-direction.

The kinetic energy is, with w indicating the velocity component in the z-direction,

$$\mathrm{KE} = \frac{1}{2}\iiint \rho[(u-c)^2 + v^2 + w^2]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \tfrac{1}{2} I. \tag{22}$$

Again

$$I = c^2 I_1 - I_2, \tag{23}$$

where

$$I_1 = \iiint \rho\left(1 - \frac{u}{c}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad I_2 = \iiint \rho(cu - q^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{24}$$

with

$$q^2 = u^2 + v^2 + w^2.$$

Using the stream functions ψ and χ, we have

$$(\phi_x, \phi_y, \phi_z) = (u, v, w) = \operatorname{grad}\psi \times \operatorname{grad}\chi. \tag{25}$$

Thus

$$q^2 = \frac{\partial(\phi, \psi, \chi)}{\partial(x, y, z)} = J,$$

$$\frac{u}{q^2} = \frac{1}{J}\frac{\partial(\psi, \chi)}{\partial(y, z)} = \frac{\partial x}{\partial \phi},$$

and again

$$\begin{aligned} I_2 &= \iiint \rho\left(\frac{cu}{q^2} - 1\right)\mathrm{d}\phi\,\mathrm{d}\psi\,\mathrm{d}\chi \\ &= \iiint \rho\left(c\frac{\partial x}{\partial \phi} - 1\right)\mathrm{d}\phi\,\mathrm{d}\psi\,\mathrm{d}\chi = 0, \end{aligned} \tag{26}$$

on account of (21). The result (26) is new, for Levi-Civita (1912, 1921) proved (17) only for two-dimensional water waves.

The same argument shows

$$I_1 = m_d, \tag{27}$$

where m_a is the drift mass per cell. The momentum M per cell is again

$$M = -cm_d, \tag{28}$$

and (22), (23) and (26) show that

$$\mathrm{KE} = \tfrac{1}{2}m_a c^2. \tag{29}$$

Thus the same results hold for three-dimensional periodic waves.

4. The special case of linear water waves

For two-dimensional linear water waves we have (x and y now for fixed frame)

$$\phi'_{xx} + \phi'_{yy} = 0. \tag{30}$$

The boundary condition at the bottom is

$$\phi'_y = 0 \quad \text{at} \quad y = -h, \tag{31}$$

and the free-surface conditions are

$$\eta_t = \phi'_y, \quad \phi'_t + \left(g + \frac{\sigma k^2}{\rho}\right)\eta = 0, \tag{32}$$

where k is the wavenumber $2\pi/\lambda$, λ being the wavelength, η is the free-surface displacement from $y = 0$, and σ is the surface tension.

The solution, for waves going left, contains the factor $(x + ct)$, so that

$$\frac{\partial}{\partial t} = c\frac{\partial}{\partial x}. \tag{33}$$

We shall now show the equipartition of energy when the effect of surface tension is included. Multiplying (30) by ϕ' and integrating by parts, using (30) to (33), we have

$$\frac{2}{\rho}(\mathrm{KE}) = \iint (\phi_x'^2 + \phi_y'^2)\,\mathrm{d}x\,\mathrm{d}y = \int_0^\lambda \phi'\,\phi'_y\,\mathrm{d}x = \int_0^\lambda \phi'\eta_t\,\mathrm{d}x = -\int_0^\lambda \eta\phi'_t\,\mathrm{d}x. \tag{34}$$

Thus, upon using the second equation in (32), we have (a = amplitude of η)

$$\frac{2}{\rho}(\mathrm{KE}) = \frac{ga^2\lambda}{2} + \frac{\sigma a^2 k^2 \lambda}{2\rho}. \tag{35}$$

But the surface energy per wavelength is

$$\mathrm{SE} = \sigma\left[\int_0^\lambda (1+\eta_x^2)^{1/2}\,\mathrm{d}x - \lambda\right] = \tfrac{1}{4}\sigma a^2 k^2 \lambda, \tag{36}$$

upon neglecting terms of $O(a^4)$. Multiplying (35) by $\rho/2$, we see that

$$\mathrm{KE} = \mathrm{PE} + \mathrm{SE}, \tag{37}$$

and (18) becomes

$$E = m_a c^2, \tag{38}$$

which resembles a famous formula in physics.

The corresponding result for three-dimensional periodic linear waves can be as easily obtained. We note that h can be infinite. For waves of finite amplitude, (37) is no longer true, and the difference between its two sides may become significant. See Lighthill (1978, figure 113).

Before turning to sound waves, we note that the result (14) is true even for solitary waves, but (18) depends on the vanishing of I_2, which is not zero for solitary waves. Thus (18) does not hold for such waves. We note, however that I_2 is equal to $\rho h[\phi'(\infty)-\phi'(-\infty)]$, and therefore to the integral of u' between $y=-h$ and $y=0$. Thus, I_2 is exactly the momentum of the fluid in that strip. Hence, instead of saying 2 (KE) is equal to $-cM$ (for a solitary wave going to the left), (8) simply says that 2 (KE) $=-c$ times the momentum of the fluid above $y=0$. This result for solitary waves escaped the attention of both McCowan (1891) and Longuet-Higgins (1974), who re-derived McCowan's result, which is (B) of Longuet-Higgins (1974). In it, Longuet-Higgins identified $\phi'(\infty)-\phi'(\infty)$ only as a circulation C, thus narrowly missed its momentum interpretation. We note also that in a paper on periodic waves (Longuet-Higgins 1975), the kinematic significance of $-M/c$, which in our notation is the drift mass per time period (for waves going to the left), also escaped his notice.

Note also that if we abandon the requirement of irrotationality, the derivation of (14) remains intact, and can be applied to internal water waves. The amplitude for these waves, with vorticity in general, can be finite. But (17) and (18) are lost. Similarly, for three-dimensional internal waves, (27) and (28) stand, but I_2 is no longer zero in general, and (29) is lost.

5. Sound waves

Solutions for linear sound waves are well known. For nonlinear sound waves propagating uni-directionally, the method of characteristics can be applied. But no solutions for periodic nonlinear sound waves seem to be known. However, their existence can hardly be doubted, since sound waves are a daily experience and although their amplitude is small any measurable amplitude is finite, and the governing equations are nonlinear. The question is only how large the amplitude can be. One fears that the characteristics may converge in the compression regions to form caustics, indicating impending shock waves. But in periodic sound waves regions of rarefaction follow those of compression, and before caustics can form in the compression regions they may diverge in the rarefaction ones, thus preventing caustics from ever forming. We shall assume the existence of nonlinear periodic sound waves and proceed on that assumption, being quite assured that the results will at least apply to linear periodic sound waves.

The analysis follows closely that for water waves. We assume the fluid to be homentropic and the flow to be irrotational and periodic, with the waves going left with speed c. For two-dimensional waves, equations (1)–(14) remain intact, but ρ is now variable, and the domain of integration D is now bounded by $x=0$, $x=\lambda$, $y=0$, and $y=1$ (say). In particular, we have (8), (13) and (14). To show that $I_2=0$, we recall that

$$(\rho u, \rho v) = \rho_0(\psi_y, -\psi_x), \tag{39}$$

where ψ is the stream function and ρ_0 a reference density. Since (5) still stands, we have

$$\rho q^2 = \rho_0 J, \quad J = \frac{\partial(\phi, \psi)}{\partial(x, y)}, \tag{40}$$

where q^2 is defined by (10).

Then, with c now indicating the sound speed,

$$I_2 = \iint \rho_0 \left(\frac{cu}{q^2} - 1\right) \mathrm{d}\phi \, \mathrm{d}\psi. \tag{41}$$

Since

$$\frac{u}{q^2} = \frac{\partial x}{\partial \phi}, \tag{42}$$

we have, on account of (4),

$$I_2 = 0, \tag{43}$$

which was not known before. Then (8) and (13) give the new results

$$2\,\mathrm{KE} = m_d\, c^2 = -Mc. \tag{44}$$

The three-dimensional case can be similarly treated. Again we take a wave cell. Instead of (25) we have

$$(\phi_x, \phi_y, \phi_y) = (u, v, w) = \frac{\rho_0}{\rho} \operatorname{grad} \psi \times \operatorname{grad} \chi, \tag{45}$$

and the rest follows, *mutatis mutandis*, the development for water waves. Again we have (27), (28) and (29), now for three-dimensional sound waves. These results are entirely new.

Clearly, the method employed here is that used by Yih (1995). But the problem treated now is physically different, since there is no body moving in the fluid now, as assumed in Yih (1995).

For linear sound waves, Lighthill (1978, p. 13) showed that the kinetic energy and the potential energy per wavelength are equal. Hence (38) again stands, now for linear sound waves.

6. The reality and significance of wave momentum

Now we shall attempt to reconcile our position (and Levi-Civita's and Rayleigh's) on wave momentum with McIntyre's (and Brillouin's). McIntyre gave the simple example of sound waves produced by a solid plane oscillating about a fixed mean position in a tube containing air (say), with a perfect absorber, also oscillating about a fixed mean position somewhere down the tube – 'perfect' in the sense that no part of the waves is reflected. Then of course the air is going nowhere in the mean, and the mean momentum, or momentum per wavelength, must be zero, as well as the drift mass. This is a compelling example.

But does that example represent the usual mathematical solution for (linear) sound waves? Are there other examples of physically produced sound waves that can have a non-zero momentum per wavelength?

For ease of exposition, let the meaning of symbols m_d and M be as in §5. Now take the well-known linear solution for sound waves, and compute the momentum and drift mass per wavelength. We get M and m_d, respectively, which are not zero. This means that McIntyre's example is not represented by the usual mathematical solution for linear sound waves, which we shall call S_1. This answers the first question posed above – in the negative. The solution, which we shall call S_2, that represents sound waves in McIntyre's example is $S_1 + S_3$, where S_3 corresponds to a uniform flow with the mass-transport speed (defined as $m_d/\rho T$, T being the period) in the direction opposite to that of wave propagation. This obviously will allow two solid plates to oscillate about fixed positions, and S_2 obviously remains a solution of the governing equations, amounting to a solution for sound waves propagating against a slight wind. The only effect on a sensor fixed on the wall of the tube would be a slightly reduced frequency (from that of the oscillating plates), due to the Doppler effect.

The question then is whether all physically produced sound waves in a tube are represented by McIntyre's example. This is the second question posed above. Immediately and naturally, wind instruments come to mind. In none of these can a mass transport be ruled out. Wind is blown into them when they are played. This is how they acquired their group name. Even if this is less obvious in the case of the flute, the opening at the mouth pad would allow air to be entrained into the tube. Entrainment can be expected if, for another instance, a tube open at its ends is held at a distance from a sound emitter placed near one end of the tube.

Even in McIntyre's example, the kinetic energy is related to the difference in frequency between the sound emitter and the sound sensor in the tube, and that difference is intimately related to our m_d or M. So that momentum which is called a 'myth' by McIntyre is intimately and simply related to measurable quantities (KE and the Doppler shift), which are real.

Let us now turn to water waves, for which the reconciliation with McIntyre's contention is even more interesting. McIntyre considered, in effect, two water-wave trains created by a wave maker (e.g. by a pressure oscillatorily acting on a part of the water surface for a period of time), one going to the right and the other to the left. It is sufficient to take one of them, say the one going to the right, after sufficient time has elapsed for the two trains to be far apart, and for the transient effects to become negligible. This seems to be McIntyre's intent when he presented his figure 2, in which he took into account the rate of mass transport (à la Ursell and Longuet-Higgins!) by a source at the head of the wave group and a sink at its tail, thus creating an irrotational back flow. The weakness of this flow allows him to regard the water surface as rigid as far as this flow is concerned. The concentrated source and sink can be replaced by source and sink distributions, but for a discussion of McIntyre's claim there is no point for that elaboration. The back flow obviously and necessarily (as a result of intention) cancels the drift mass m_d in §2, and therefore the momentum M. Again it is a compelling example, for a wave train of finite length is more realistic than one that is infinitely long, for which the back flow is obscured.

But, does it represent all water waves in all circumstances? Imagine a canal connecting two very large reservoirs, with equal surface elevations when at rest. Then imagine waves approaching one end of the canal. Part of the waves will enter the canal and through it the other reservoir. Can one say there is no wave momentum and no mass entrainment into and transport through the canal? This example is similar to the one for sound waves entering a tube from a source outside of it.

More importantly, even in McIntyre's example wave momentum cannot be dismissed as a mere 'myth'. The strength of the source or sink in figure 2 of McIntyre (1981) is the rate of mass transport (divided by the density ρ) calculated by Ursell (1953) and Longuet-Higgins (1953) for linear water waves, or just our $m_d/\rho T$ for linear or nonlinear water waves. It is independent of the length of the wave group. For a fixed wavelength, when the depth of the ocean and the length of the group both become larger and larger, the velocity of the back flow away from the ends of the group becomes weaker and weaker, and approaches zero as a limit. If one calculates the drift mass as in §2, integrating to a depth equal to several (say 5) wavelengths, one obtains a value very close to m_d or M/c (for waves going to the right). Doubling or tripling the depth of integration makes a negligible difference. Of course, integration all the way to the bottom would indeed give the value zero. But the definite value very close to m_d or M/c obtained by integrating part of the way cannot be dismissed as a myth. It is very relevant to the mean force exerted by water on a floating body, or to the spread of pollutants or solutes in the upper layer (wave layer) of the ocean.

This m_d (or M) held in a widespread thin counter current is reminiscent of the value of Darwin's drift mass, when it is calculated by integrating longitudinally first. That value is equal to the added mass, which we shall denote by m_a. When it is calculated by integrating transversely first (in planes $x = \pm x_0, x_0$ large) one gets a value different from m_a. The difference is, however, spread over a large distance or area, and if one carries out the transverse integration over a distance or area 5–10 times the body size one obtains a value very close to m_a. Again this drift mass (nearly) equal to m_a is held in a weak widely spread back flow. Whereas this back flow merely reduces the drift mass to zero in McIntyre's example, it makes the total calculated drift mass negative in Darwin's problem, turning the drift into a reflux. This little difference apart, the similarity of the situations is interesting, especially the significance, in both instances, of a quantity (m_a, or our m_d or M) embedded in a widely spread weak current, which does not quite manage to hide it.

Finally, it is appropriate here to mention that for two-dimensional water waves propagating in the x-direction with speed c, the rate of flow of energy (KE + PE) through a section with constant x is equal to c times the flux of moment in the x-direction through the same section. See Wehausen & Laitone (1960, equation (8.5)). But here the drift mass plays no recognizable role, except in the special case of linear two-dimensional waves in very deep water, provided the effect of surface tension is neglected. In this case it is known that the energy flux through a section of constant x per time period T is exactly one half the energy E in one wavelength, because the group velocity is half the phase velocity. Since the energy is equally partitioned into its kinetic and potential parts, the energy flux in T is exactly the kinetic energy in one wavelength, given by (18). There is a corresponding result for one-dimensional linear sound waves. Since these are non-dispersive, so that the group velocity is equal to the phase velocity, the energy flux in one time period is exactly the E given by (38), with c now indicating the sound speed. So in these two special cases at least, the drift mass is significant even in the rate of energy flow. Energy flux and momentum flux in water waves have been treated by Starr (1947), Platzman (1947), and Starr & Platzman (1948). See Wehausen & Laitone (1960, pp. 718–723).

REFERENCES

Benjamin, T. B. 1995 Verification of the Benjamin–Lighthill conjecture about steady water waves. *J. Fluid Mech.* **295**, 337–356.

Chen, B. & Saffman, P. G. 1980 Numerical evidence for the existence of new types of gravity waves of permanent form on deep water. *Stud. Appl. Maths* **62**, 1–21.

Darwin, C. 1953 Note on hydrodynamics. *Proc. Camb. Phil. Soc.* **49**, 342–354.

Eames, I., Belcher, S. E. & Hunt, J. C. R. 1994 Drift, partial drift, and Darwin's proposition. *J. Fluid Mech.* **275**, 201–223.

Jeffreys, H. & Jeffreys, B. S. 1965 *Methods of Mathematical Physics*. Cambridge University Press.

Levi-Civita, T. 1912 Sulle onde di canale. *Atti Accad. Lincei, Rend. Cl. Sci. Fis. Mat. Nat.* (5) **21**, 1° sem., 3–14.

Levi–Civita, T. 1921 Qüestions de mecànica clàssica i relativista. *Conferències Donades el Gener de 1921*. Barcelona: Institut d'Estudis Catalans 1922.

Lighthill, M. J. 1978 *Waves in Fluids*. Cambridge University Press.

Longuet-Higgins, M. S. 1953 Mass transport in water waves. *Phil. Trans. R. Soc. Lond.* A **245**, 535–581.

LONGUET-HIGGINS, M. S. 1974 On the mass, momentum, energy and circulation of a solitary wave. *Proc. R. Soc. Lond.* A **337**, 1–13.

LONGUET-HIGGINS, M. S. 1975 Integral properties of periodic gravity waves of finite amplitude. *Proc. R. Soc. Lond.* A **342**, 157–174.

MCCOWAN, J. 1891 On the solitary wave. *Phil. Mag.* (5)**1**, 257–279.

MCINTYRE, M. E. 1981 On the 'wave momentum myth. *J. Fluid Mech.* **106**, 331–347.

PLATZMAN, G. W. 1947 The partition of energy in periodic irrotational waves on the surface of deep water. *J. Mar. Res.* **6**, 194–202.

STARR, V. P. 1947 A momentum integral for surface waves in deep water. *J. Mar. Res.* **6**, 126–135.

STARR, V. P. & PLATZMAN, G. W. 1948 The transmission of energy by gravity waves of finite height. *J. Mar. Res.* **7**, 229–238.

STOKES, G. G. 1947 On the theory of oscillatory waves. *Trans. Camb. Phil. Soc.* **8**, 441–455.

STRUIK, D. J. 1926 Détermination rigoureuse des ondes irrotationelles permanentes dans un canal à profondeur finie. *Math. Ann.* **95**, 595–634.

URSELL, F. 1953 Mass transport in gravity waves. *Proc. Camb. Phil. Soc.* **49**, 145–150.

WEHAUSEN, J. V. & LAITONE, E. V. 1960 Surface waves. In *Handbuch der Physik*, 9 (ed. C. Truesdell). Springer.

YIH, C.-S. 1985 New derivations of Darwin's Theorem. *J. Fluid Mech.* **152**, 163–172.

YIH, C.-S. 1995 Kinetic-energy mass, momentum mass, and drift mass in steady irrotational subsonic flows. *J. Fluid Mech.* **297**, 29–36.